

4-point Fermat location problems revisited. New proofs and extensions of old results

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January 26 2005

Abstract

What is the point at which the sum of (euclidean) distances to four fixed points in the plane is minimised? This extension of the celebrated location question of Fermat about three points was solved by Fagnano and others around 1750, giving the following simple geometric answer: when the fixed points form a convex quadrangle it is the intersection point of both diagonals, and otherwise it is the fixed point in the triangle formed by the three other fixed points.

We show that the first case extends and generalizes to general metric spaces, while the second case extends to any planar norm, any ellipsoidal norm in higher dimensional spaces, and to the sphere.

1 Introduction

Around four centuries ago, Fermat [8, p153] asked to find a point minimising the sum of distances to three fixed points in the (euclidean) plane, thereby unknowingly initiating the family of minimum location problems. Although Fermat's original question was fully answered within the same century by many scholars, including illustrious names like Torricelli, Ricci, Cavalieri, Viviani, Renieri, (see the authoritative survey [13]), it remains still vigorously studied under many different points of view and extensions, see e.g. [12], [4], [13]. The simplest extension to four instead of three fixed points, which we will call the 4-point Fermat location problem, has the following well-known complete answer (using the classification of [13]).

Theorem 1 *The sum of euclidean distances to four fixed points p, q, r, s in the plane is minimised at*

(Floating Case) *the point of intersection of the diagonals, when the fixed points form a convex quadrangle,*

*The last version of this paper was written while visiting the Mathematics Institute of the Technische Universität Graz, Austria, whose support is gratefully acknowledged. The author thanks an unknown referee for several helpful indications and references.

(Absorbed Case) otherwise the fixed point which belongs to the (closed) triangle formed by the three other fixed points.

Proof

(Floating Case) The following extremely simple proof (notations adapted) involving only the triangle inequality goes back to Fagnano in 1775 [7, *Solution per simplicem Geometriam. (Fig. XIV.)*, p295], as opposed to the one and half page proof involving differential calculus (*Infinitorum Methodum*) [7, p293–295]:

Let $u \in [p, r] \cap [q, s]$ and $x \in \mathbb{R}^2$. Note that $d(p, r)(= d(p, u) + d(u, r)) \leq d(p, x) + d(x, r)$; and similarly $d(q, s)(= d(q, u) + d(u, s)) \leq d(q, x) + d(x, s)$, therefore $d(p, u) + d(r, u) + d(q, u) + d(s, u) \leq d(p, x) + d(r, x) + d(q, x) + d(s, x)$

(Absorbed Case) This case was apparently not considered by Fagnano [7], (as erroneously suggested in [3, p104]) probably because no simple geometric proof was available. The following proof slightly shortens the arguments in [3, p111]:

Let $s \in \text{conv}(p, q, r)$, and consider any point x in the plane. In case $x \notin \text{conv}(p, q, r)$ we may separate x strictly from $\text{conv}(p, q, r)$ by a line L . The orthogonal projection of x on L is then strictly closer to each p, q, r, s than x , and hence x cannot be optimal. It is therefore sufficient to show that s yields at least the same objective value as any $x \in \text{conv}(p, q, r)$. For any such x an adequate renaming of p, q, r yields $s \in \text{conv}(p, q, x)$, and hence the halfline ps intersects $[x, q]$ in a point y . Then $y \in [x, q]$ and $s \in [py]$ and it follows that $d(p, s) + d(q, s) \leq d(p, s) + d(s, y) + d(y, q) = d(p, y) + d(q, x) - d(x, y) \leq d(p, x) + d(q, x)$. Summing with the triangle inequality $d(r, s) \leq d(r, x) + d(x, s)$ and the fact $d(s, s) = 0$ we obtain the result.

□

In this note we offer new proofs for each of both cases which allow to generalise these results for other distance measures like general metrics and norms, and going beyond several such extensions already known.

2 Extensions of the floating case

The simplicity of the argument of Fagnano for showing the absorbed case strongly suggests that the result should extend to much more general situations. We start by stating a trivial lemma, which turns out to be a basic result needed in this section.

Lemma 2 *Let the objective function of an optimisation problem*

$$(P) : \min\{ f(x) \mid x \in S \}$$

be a positively weighted sum of objectives f_i ($i \in I$), i.e. $f(x) = \sum_{i \in I} w_i f_i(x)$ with all $w_i > 0$. If the subproblems

$$(P_i) : \min\{ f_i(x) \mid x \in S \}$$

($i \in I$) have a common optimal solution $x^* \in S$, then the optimal solution set $\text{opt}(P)$ of (P) equals the intersection of the optimal solution sets $\text{opt}(P_i)$ of all (P_i) .

Proof For any $y \notin \bigcap_{i \in I} \text{opt}(P_i)$ and any $x \in \bigcap_{i \in I} \text{opt}(P_i)$ we have $f_i(y) \geq f_i(x) = f_i(x^*)$ for all $i \in I$, with at least one strict inequality. Therefore $f(y) = \sum_{i \in I} w_i f_i(y) > \sum_{i \in I} w_i f_i(x) = f(x) = f(x^*)$, and the conclusion follows. \square

2.1 General metric distance

Let (X, d) be any metric space. The *metric segment* between two points $a, b \in X$ was defined by Menger [16] as the set of points at which the triangle inequality on a, b is in fact an equality:

$$[a, b]_d \stackrel{\text{def}}{=} \{ c \in X \mid d(a, c) + d(c, b) = d(a, b) \}$$

Note that because of the triangle inequality for d this definition implies that $x \notin [a, b]_d$ is equivalent to $d(a, x) + d(x, b) > d(a, b)$.

For any finite set $P \subset X$, consider the (generalized) Fermat-problem to minimise the sum of distances function

$$F_P(x) \stackrel{\text{def}}{=} \sum_{p \in P} d(x, p)$$

The set of minimisers of F_P will be called the *median set* of P , denoted by $\text{med}(P)$.

Lemma 3 *2-point Fermat problem.*

For any pair a, b in a metric space X, d we have

$$[a, b]_d = \text{med}(\{ a, b \})$$

Proof For any $c \in [a, b]_d$ and $x \notin [a, b]_d$ we have

$$d(c, a) + d(c, b) = d(a, c) + d(c, b) = d(a, b) < d(a, x) + d(x, b) = d(x, a) + d(x, b)$$

\square

The following theorem was obtained in [11] (slightly extended to weighted situations), and also in [15] for normed spaces.

Theorem 4 *If P can be partitioned into a set of pairs $\{ p_i, p'_i \}$ ($i \in I$), such that*

$$\bigcap_{i \in I} [p_i, p'_i]_d \neq \emptyset$$

then this intersection is exactly the median set of P .

Proof This is a direct application of lemmas 2 and 3. \square In particular

Corollary 5 *4-point Fermat problem.*

For any 4-point set $P = \{p, q, r, s\}$ in a metric space X, d with $[p, q]_d \cap [r, s]_d \neq \emptyset$ we have

$$\text{med}(P) = [p, q]_d \cap [r, s]_d$$

Proof

□

This gives the full answer to the 4-point Fermat problem, provided we may construct these metric segments. In the next subsections we consider a number of cases where this applies.

2.2 Network

On an (undirected) network with shortest path distance the metric segment $[a, b]_d$ for two points a and b is clearly the union of all shortest paths between a and b .

Theorem 6 *Let $P = \{p, q, r, s\}$ be a set of four points on an undirected network. If some shortest path connecting p and q meets a shortest path connecting r and s , then the median set of P consists of all points common to any such pair of shortest paths.*

Note that when all four fixed points are nodes, it follows that the median set in this case always consists of a union of edges and vertices, and thus always contains a vertex, in accordance with Hakimi [9].

2.3 Sphere distance

In [14], Fagnano's result was obtained for four points on a same hemisphere, by way of an analytical reasoning. Lemma 2 allows to obtain a much more complete answer on the sphere, as follows. We call two points on the sphere *diametrical* when they are the extreme points of a sphere's diameter.

On the sphere with great circle distance one easily sees that the metric segment between any two non-diametrical points is the smallest great circle arc connecting them. For two diametrical points the metric segment is the whole sphere.

Therefore we obtain following floating case on the sphere

Theorem 7 *Consider the Fermat problem on the sphere with 4 fixed points.*

- *If these are two by two diametrical, then the median set is the whole sphere.*
- *If there is one pair of diametrical fixed points, then the median set consists of the smaller great circle arc connecting the two other fixed points.*
- *If there are no diametrical points, but the fixed points come in two pairs, whose smaller connecting great circle arcs meet, then the median set is exactly the intersection between these two circle arcs. This median set can thus be either a great circle arc, if all fixed points lie on a same great circle, or a single point otherwise.*

2.4 Norm distance

Evidently in the planar euclidean distance case, metric segments are simply line-segments, the 'diagonals', which, if they meet, always meet at a single point. This is the classical floating case result of Fagnano. But this result is valid more generally, and can be made much more precise in case of norm distances.

Consider any Minkowski norm ν with unit ball $B \subset \mathbb{R}^n$, and let d_ν be the metric derived from ν by $d_\nu(x, y) \stackrel{\text{def}}{=} \nu(y - x)$. B is then a convex set, symmetric with respect to the origin. For an in depth treatment of the geometry of Minkowski spaces see [23]. A complete characterization of the metric segments for such metric spaces may either be obtained by way of duality arguments, see e.g. [15], or by the following geometrical construction fully detailed in [2, p7–12]

For any b on B 's boundary, i.e. with $\nu(b) = 1$, the face $B(b)$ of B at b is defined as follows:

$$B(b) \stackrel{\text{def}}{=} \{ a, a' \in B \mid b \in]a, a'[\} \cup \{ b \} \quad (1)$$

i.e. the largest subset of B 's boundary for which b belongs to its relative interior. Evidently, $B(b) = \{ b \}$ if and only if b is an extreme point of B , i.e. b does not lie on any open segment of B . For example, when ν is a round norm, i.e. with strictly convex B , like any ℓ_p norm with $1 < p < +\infty$, e.g. the euclidean norm ($p = 2$), then $B(b) = \{ b \}$ for any b with $\nu(b) = 1$. For block norms, i.e. with a polyhedral unit ball B , the extremality face at b is the smallest face of B to which b belongs; e.g. for the rectangular (or Manhattan) norm ℓ_1 and $b = (0, 1)$ we have $B(b) = \{ b \}$, but for $b = (0.8, 0.2)$ we have $B(b) = [(0, 1), (1, 0)]$.

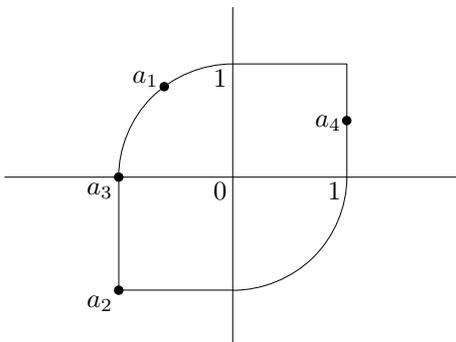


Figure 1: An example of a norm

A somewhat less trivial example is given by the ball shown in figure 1. The faces at the given points are $B(a_i) = \{ a_i \}$ for $i = 1, 2, 3$, while $B(a_4) = [(1, 0), (1, 1)]$.

For any $x \neq 0$, we define further its ν -linearity cone as

$$L(x) \stackrel{\text{def}}{=} \mathbb{R}^+ B\left(\frac{x}{\nu(x)}\right) = \{ \lambda a \mid \lambda \geq 0, a \in B\left(\frac{x}{\nu(x)}\right) \}$$

which is always a closed convex cone. Note also that, because of symmetry of ν we always have $L(x) = -L(-x)$.

For the euclidean norm the ℓ_2 -linearity cone $L(x)$ is always the halfline from the origin through x , because all boundary points of the unit circle are extreme

points. For the rectangular norm the ℓ_1 -linearity cone $L(x)$ is the halfline from the origin through x for any x on some axis, but when x is interior to a quadrant $L(x)$ is this whole (closed) quadrant. For the norm shown in figure 1, $L(x)$ is the halfline from the origin through $x = (x_1, x_2)$ as soon as $x_1 x_2 \leq 0$, but $L(a_4) = \{ x = (x_1, x_2) \mid 0 \leq x_2 \leq x_1 \}$.

For the metric d derived from ν the metric segments are constructed by

$$[x, y]_d = (x + L(y - x)) \cap (y + L(x - y)) \quad (2)$$

This always contains the segment $[x, y]$ while when ν is strictly convex we always have $[x, y]_d = [x, y]$.

Using these facts together with theorem 4 we obtain

Corollary 8 *If P forms a convex quadrangle, then the median set is the intersection of the metric segments on the diagonal pairs of P .*

Fagnano's floating case result holds for any norm distance in \mathbb{R}^n . The intersection point of the diagonals (if it exists) is the unique minimiser of F_P if and only if the direction of both diagonals define extreme points of the norm's unit ball.

The two last parts of this result are given in [3, p110].

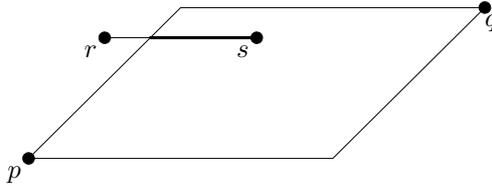


Figure 2: A 4-point Fermat problem with the norm of figure 1

But corollary 5 is often even stronger. Figure 2 shows a 4-point Fermat problem with the norm of figure 1. Observe that no diagonals meet, so strictly speaking Fagnano's result does not apply. However, the depicted metric segments $[p, q]_\nu$ (parallelogram) and $[r, s]_\nu$ (horizontal segment) do meet, so by corollary 5 their intersection (thick segment) is the median set.

3 Extensions and limits of the absorbed case

The absorbed case result considers points belonging to the (closed) triangle formed by the other points. Here the notion of convex hull is invoked, a concept which, in its classical form, calls for a vector space setting. Therefore we start by considering a real vector space \mathbb{R}^n equipped with a norm.

3.1 Norm distance

Let us first recall the notion of weakly efficient point in the context of location theory. A point x is weakly efficient with respect to the set of points A and distance measure d , when there exists no other point y such that $d(a, y) < d(a, x)$

for all $a \in A$. The set of all weakly efficient points w.r.t. A , denoted by $WE(A)$ (the distance measure having been fixed), has been studied by many authors, in particular for norm distances in location theory (see e.g. [6]), but also in the much more general setting of convex analysis (see e.g. [20]).

We will also need the following simple lemma.

Lemma 9 *Let ν be any norm on \mathbb{R}^n . If the origin 0 lies in the convex hull of k nonzero vectors of ν 's unit ball, their sum always has ν -norm $\leq k - 2$.*

Proof Let $p_i \in \mathbb{R}^n$ ($i = 1, \dots, k$) with $0 < \nu(p_i) \leq 1$ and $0 \in \text{conv}(p_1, \dots, p_k)$. This means there exist $\lambda_i \geq 0$ for $i = 1, \dots, k$, with $\sum_{i=1}^k \lambda_i = 1$ and $\sum_{i=1}^k \lambda_i p_i = 0$. After a suitable renumbering we have $\lambda_1 = \max_{i=1, \dots, k} \lambda_i$, hence $\lambda_1 > 0$ and $1 - \frac{\lambda_i}{\lambda_1} \geq 0$ for all i . From the first inequality it follows that $\lambda_1 = \nu(\lambda_1 p_1) = \nu(-\sum_{i=2}^k \lambda_i p_i) \leq \sum_{i=2}^k \lambda_i \nu(p_i) = \sum_{i=2}^k \lambda_i$. We then have

$$\begin{aligned}
\nu\left(\sum_{i=1}^k p_i\right) &= \nu\left(p_1 + \sum_{i=2}^k p_i\right) \\
&= \nu\left(\sum_{i=2}^k \left(1 - \frac{\lambda_i}{\lambda_1}\right) p_i\right) \\
&\leq \sum_{i=2}^k \left(1 - \frac{\lambda_i}{\lambda_1}\right) \nu(p_i) \\
&\leq \sum_{i=2}^k \left(1 - \frac{\lambda_i}{\lambda_1}\right) \\
&= k - 1 - \frac{\sum_{i=2}^k \lambda_i}{\lambda_1} \\
&\leq k - 1 - \frac{\lambda_1}{\lambda_1} \\
&= k - 2
\end{aligned}$$

□

Theorem 10 *Consider the Fermat problem F_P where $P = \{ p, q, r, s \} \subset \mathbb{R}^n$, and distance d derived from a norm ν . If $s \in WE(p, q, r)$ then s minimises F_P .*

Proof Since F_P is a convex function it is sufficient to show that 0 is a subgradient of F_P at s (see [10] for the general theory of subgradients and [18] for their application in Fermat-Weber problems). The subdifferential of F_P at s is obtained as follows: $\partial F_P(s) = \sum_{a \in P} \partial d_a(s)$, where $d_a(x) = d(a, x) = \nu(x - a)$. Now $\partial d_s(s) = \{ u \in \mathbb{R}^d \mid \nu^0(u) \leq 1 \}$, the unit ball for the dual norm ν^0 of ν . It follows that optimality of s is equivalent to finding subgradients $\pi \in \partial d_p(s)$, $\xi \in \partial d_q(s)$ and $\rho \in \partial d_r(s)$ such that $\nu^0(-\pi - \xi - \rho) \leq 1$.

But $s \in WE(P')$, where $P' \stackrel{\text{def}}{=} \{ p, q, r \}$. It follows (see e.g. [20]) that $0 \in \text{conv}(\cup_{a \in P'} \partial d_a(s))$, in other words there exist $\pi \in \partial d_p(s)$, $\xi \in \partial d_q(s)$ and $\rho \in \partial d_r(s)$ such that $0 \in \text{conv}(\pi, \xi, \rho)$. But any subgradient to a norm has dual

norm ≤ 1 , hence we also have $\nu^0(\pi) \leq 1$, $\nu^0(\xi) \leq 1$ and $\nu^0(\rho) \leq 1$, and by application of lemma 9 we have $\nu^0(-\pi - \xi - \rho) = \nu^0(\pi + \xi + \rho) \leq 1$. \square

A norm on \mathbb{R}^n is called ellipsoidal (see [17]) if its unit ball is an ellipsoid, or, equivalently, if it is of the form $\nu(x) = \ell_2(Ax)$ where A is some regular linear transformation of \mathbb{R}^n , i.e. if it defines an inner product space.

We obtain the following corollary, the first part of which was already obtained by Cieslik [3, p.110] and Swanepoel [21].

Corollary 11 *The absorbed case result holds for any norm in \mathbb{R}^2 , or any ellipsoidal norm in \mathbb{R}^n .*

Proof With the same notation as in previous theorem we have $s \in \text{conv}(P')$. It is known that for any norm in \mathbb{R}^2 (see [22]) or any ellipsoidal norm in \mathbb{R}^n (see [17]) we have $s \in WE(P')$. The result then follows from previous theorem. \square

For dimension > 2 and non-ellipsoidal norms, $WE(P')$ may contain points outside $\text{conv}(P')$, see e.g. the example for the ℓ_p -norm ($p \neq 2$) given in [17]. Theorem 10 would still apply in such cases.

It is well known that for round norms (i.e. with strictly convex unit balls) any Fermat-Weber problem with non collinear fixed points has a unique optimal solution. It follows that for planar round norms corollaries 8 and 11 give a complete answer to the 4-point Fermat problem.

For planar norms which are not round we may obtain a full solution to the 4-point Fermat problem by observing

- in the convex quadrangle case theorem 8 gives a complete solution
- in the triangle including point s case, theorem 11 tells us that s is an optimal solution, and [5] shows that the median set is an elementary convex set. We may deduce that the median set is thus the elementary convex set containing s .

In higher dimensions, however, the two cases do not cover all possibilities, so, even when restricting attention to ellipsoidal norms, the 4-Fermat problem is not yet fully solved. Even in euclidean 3-space no constructive method seems to exist, see [13] for details.

In [1] it is shown that any normed space of dimension ≥ 3 in which for any 3-point Fermat problem F_P some median exists in $\text{conv}(P)$, is necessarily ellipsoidal. In the same vein we conjecture the following somewhat more geometrical property

Conjecture 12 *If in \mathbb{R}^n ($n > 2$) equipped with a norm ν , the absorbed case result always holds, then ν is ellipsoidal.*

3.2 Metric space

In [13, p68] it is suggested that the absorbed case (and the floating case) may be proven by the triangle inequality only. However, if the triangle inequality would suffice for a proof, the absorbed case would hold for any metric defined on the plane, which is not the case, as shown by following counterexample.

Example 1 Consider the plane folded down at a right angle along its first axis, with metric inherited by embedding in the euclidean \mathbb{R}^3 . This is more formally defined as follows by the embedding

$$e : \mathbb{R}^2 \rightarrow \mathbb{R}^3 : (x_1, x_2) \mapsto \begin{cases} (x_1, x_2, 0) & \text{when } x_2 \geq 0 \\ (x_1, 0, x_2) & \text{when } x_2 < 0 \end{cases}$$

and the distance measure on \mathbb{R}^2 defined by $d(x, y) = \|e(x) - e(y)\|$.

Consider the points $p = (0, -10)$, $q = (10, 10)$, $r = (-10, 10)$ and $s = (0, 0) \in \text{conv}(p, q, r)$. Then $f(s) = d(p, s) + d(q, s) + d(r, s) + d(s, s) = 10 + 20\sqrt{2} \approx 38.3$. But at the point $x = (0, 1)$ we have $f(x) = \sqrt{101} + 2\sqrt{181} + 1 \approx 38.0 < f(s)$.

It may be observed that our proof of the absorbed case heavily relies on convexity of the norm. Also the proof in the classical planar euclidean case involved some convexity arguments, in particular linear separation between a point and the convex hull of P , and equality of the triangle inequality along a line segment. Now if convexity of the distance to fixed points is needed, according to Witzgall [24] this calls for a metric derived from a norm. Therefore we conjecture

Conjecture 13 *If in \mathbb{R}^2 equipped with a metric d , the absorbed case result always holds, then d is derived from some norm.*

3.3 Sphere distance

The stereographic projection arguments used in [14] may quite easily be extended to prove the following absorbed case result on the sphere (one may wonder why this case was not considered there, but only the floating case, which may be solved more completely using much simpler means).

Theorem 14 *If three fixed points contained in a hemisphere form a convex spherical triangle containing the fourth fixed point, then the corresponding 4-Fermat problem on the sphere is optimised at this fourth fixed point.*

3.4 Network

One might be tempted to try extending these results to networks but it is not clear by what notion the ‘triangle’ should be replaced. Simply negating the intersecting shortest path situation of theorem 6, and using as proxy for the triangle the ‘shortest path-closure’ of the fixed points, i.e. the smallest subset of the network containing all fixed points and any shortest path between any pair of points of this subset, is bound to fail, as shown by the following counterexample.

Let the network consist of K_5 , a complete 5-node graph, and call one of the nodes a , the other four being the fixed points of P . Let all edges at a have length 2, and all remaining edges length 3. Clearly all shortest paths between two nodes consist of single edges. It follows that P cannot be split into two pairs with intersecting shortest paths. Also, the shortest path closure of P consists of the (complete) subgraph $K_4(P)$ induced by P . By the well-known node-optimality theorem an optimal solution to F_P is found at some node of the network, and one easily checks that the only solution is found at a , which is not part of $K_4(P)$.

4 Not much hope in case of asymmetry

We have shown that both results of Fagnano may be extended, the first to general metric spaces, the second to planar normed spaces.

It may be seen that all arguments make extensive use of the symmetry of the distance measure. Therefore further extensions to asymmetric distance situations as discussed in [18] or in more general settings in [19] seem out of question.

First, the notion of metric segment is unrelated to Fermat-problems for asymmetric distance. The only possible extension here would be replacing it by the median set of a 2-point Fermat problem, which for asymmetric distances might contain none of the fixed points themselves. Therefore there seems to be no relation between diagonals and their possible intersection with median sets.

Second, asymmetric gauge Fermat problems may have median sets which lie totally outside the convex hull of the fixed points, so an absorbed case type result seems also out of reach.

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